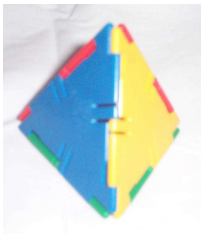
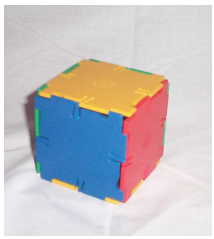
Summer Educational Enrichment in Math, 2024
Math Contest - Solutions

1. **Platonic Solids**: Name the 5 Platonic Solids and say how many faces they have. (Spelling does not matter.)



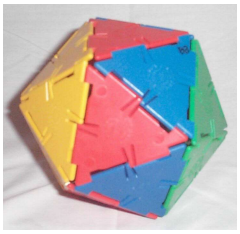
Name Tetrahedron
Faces 4



Name Hexahedron or Cube
Faces 6

Name Octahedron
Faces 8

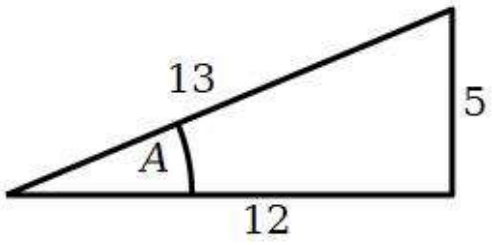
Name Dodecahedron
Faces 12



Name Icosahedron
Faces 20

2. **Trig Functions**:

For the right triangle at the right, identify the trig functions for the angle A.

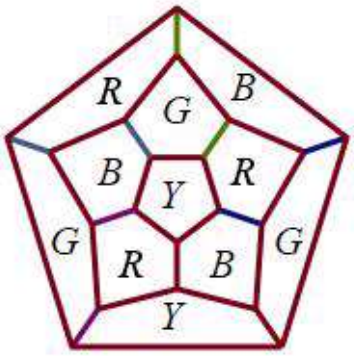


$\sin A =$_____ $\cos A =$_____

Solution:

$$\sin A = \frac{Opp}{Hyp} = \frac{5}{13} \qquad \cos A = \frac{Adj}{Hyp} = \frac{12}{13}$$

3. **Map Coloring**: The map at the right has 11 countries.



Color it with as few colors as possible.

Countries with a common edge must have different colors.

Use the abbreviations:

R=red B=blue G=green Y=yellow P=purple

Explain why you cannot do it with fewer colors.

Solution: It can be done with 4 colors. This is a posible solution.
It can't be done with 3 colors because the central yellow country is surrounded by 5 countries which can start alternating between 2 colors, blue and red, but this cannot go all the way around because 5 is odd. So there must be a 4th color, green.

4. **Euler numbers**: Consider the octahexahedron made from 6 squares and 8 triangles:

The number of faces is: $\boxed{F = 14}$

The number of vertices is: $\boxed{V = 12}$ Explain below.

The number of edges is: $\boxed{E = 24}$ Explain below.

Explain V: **Solution**: The 6 squares have 4 vertices and the 8 triangles have 3 vertices for a total of $6 \times 4 + 8 \times 3 = 48$ vertices, counting each vertex for each face, but each vertex belongs to 4 faces. So we divide to get $48/4 = 12$ vertices.

Explain E: **Solution**: The 6 squares have 4 edges and the 8 triangles have 3 edges for a total of $6 \times 4 + 8 \times 3 = 48$ edges, counting each edge for each face, but each edge belongs to 2 faces. So we divide to get $48/2 = 24$ edges.

Calculate the Euler number: **Solution**: $F + V - E = 14 + 12 - 24 = 2$

Explain how you know the Euler number before counting F, V and E?

Solution: There are no holes.

5. **Balderdice**:

a. If there are 24 dice remaining at the table, how many total dice at the table would you expect to be 5s and 1s?

Solution: In 24 random dice you would expect 4 of each of the 6 numbers. So 4 of 5s and 4 of 1s. So a total of 8.

b. If there are 24 dice remaining at the table and you managed to roll a 5 or 1 on all 3 of your own dice, how many total dice at the table would you expect to be 5s and 1s?

Solution: You have 3 dice. Of the 21 remaining dice you would expect $\frac{1}{3}$ of them to be 5s and 1s, or 7. With your 3, that's a total of 10.

6. **Matrices**: Compute the following matrix product:

$$\begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \_\_\_ \\ \_\_\_ \end{pmatrix}$$

Solution: $\begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 + 4 \cdot 2 \\ 5 \cdot 3 + 3 \cdot 2 \end{pmatrix} = \begin{pmatrix} 14 \\ 21 \end{pmatrix}$

7. **Birthday Problem**:

   a. What is the probability that Polly and Jason have different birthdays (assuming neither was born in a leap year)?

   Solution: $1 \cdot \frac{364}{365} = \frac{364}{365}$

   b. If 5 people are in a room, what is the probability that at least 2 of them have birthdays in the same month?

   Solution: The probability they all have birthdays in different months.is

   $$p = \frac{11}{12} \times \frac{10}{12} \times \frac{9}{12} \times \frac{8}{12} = \frac{11}{12} \times \frac{5}{6} \times \frac{3}{4} \times \frac{2}{3} = \frac{55}{144}$$

   So the probability they are not all different is $1 - p = 1 - \frac{55}{144} = \frac{89}{144}$.

8. **Strings**: You hold 4 strings in your hand. You tie off 2 pairs at the top and 2 pairs at the bottom. When you let go, what is the probability that the strings are all in one loop?

   Solution: The top pairings don't matter. When you connect 1 bottom string to 1 of the 3 others, 2 will allow for a single loop. So the probability is $\frac{2}{3}$. The remaining 2 strings are then tied. So the overall probability is $\frac{2}{3}$.

9. **Infinities**: TRUE OR FALSE? There are more natural numbers than there are even numbers.

   a. Circle one: TRUE or FALSE

   b. Explain your answer.

   Solution: There is a one-to -one correspondence between the natural numbers and the even numbers. Each natural number is paired with its double.

   $$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & \cdots \end{array}$$

   So these sets have the same cardinality, $\aleph_0$.

10. **Hilbert Hotel**: You run the Hilbert Hotel, which has an infinite number of rooms and is full. 2024 guests want rooms for a Math Convention. Explain how you rearrange everyone to accommodate the guests by stating which room the person in room N needs to move to.

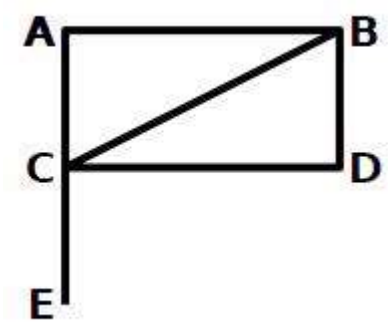
    Solution: $N \to N + 2024$

11. **Rational Tangles**: You have two ropes which were tangled using Twists (T) and Rotations (R). The tangle is assigned the rational number $\frac{-4}{3}$. Write down the list of Twists and Rotations which will untangle the ropes and the rational number assigned to each intermediate step. (There may be more blanks than you need.)

R/T		T		T		R		T		T		R		T		T	
#	$\frac{-4}{3}$	$\Rightarrow$	$\frac{-1}{3}$	$\Rightarrow$	$\frac{2}{3}$	$\Rightarrow$	$\frac{-3}{2}$	$\Rightarrow$	$\frac{-1}{2}$	$\Rightarrow$	$\frac{1}{2}$	$\Rightarrow$	-2	$\Rightarrow$	-1	$\Rightarrow$	0

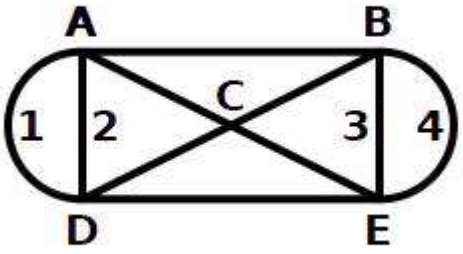
12. **Euler Paths**: For each graph below determine whether the graph is traversable (in other words, determine whether you can trace each edge of it exactly once without lifting a pencil). If it is not traversable, give a reason for your answer. If it traversable, show an Euler path ($Start \neq Finish$) or an Euler circuit ($Start = Finish$) using arrows and numbers along edges.

a.



One solution is
E C A B D C B
but it must start and end at **B** and **E**.

b.



One solution is
A B 4 E D 1 A 2 D C B 3 E C A

13. **Cryptography**:

a. Decode the following message which was encoded using a shift cipher:

I Z I V C W U Y E V I M W E V I G X E R K P I.

Solution:

EVERY S Q U A R E I S A R E C T A N G L E.

b. Decode the following message, which has 45 letters:

IESOH DGANE ORNOM NEDTS OEHLA
TNAIM LEMKI IGIEA KGDTM

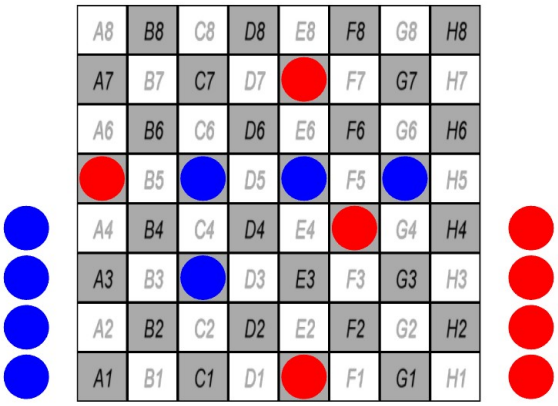
Solution: (stack them and read down):
I DO NOT LIKE GREEN EGGS AND HAM. I DO NOT LIKE THEM, SAM-I-AM.

14. **Pop-Tac-Toe**: It is Blue's turn.
Can Blue win on this turn?
Circle one:

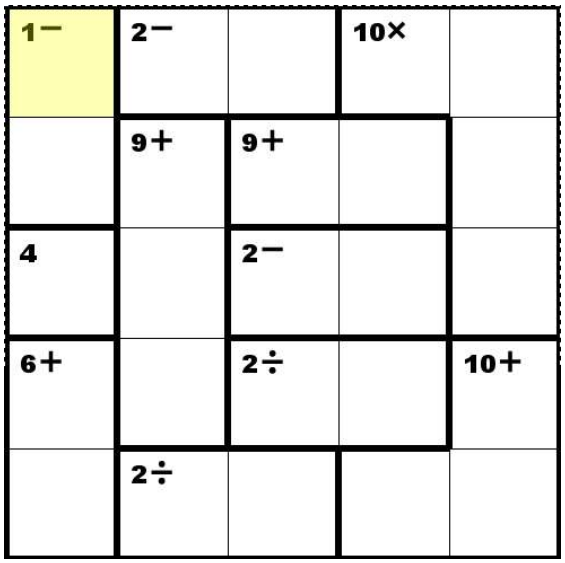
Yes No

If so, which square should Blue play on?
If there is more than one answer,
just list one winning play.

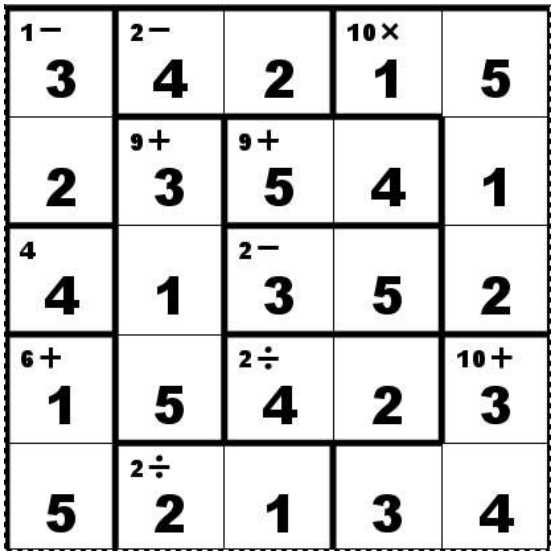
Solution: Play B6



15. **Kenken**: Solve the Kenken:



Solution:



16. Solve the cryptogram:

DKCZW VM UDT KIT, DKCZW VM CGHD,

SKJDH K SKI ODKZVOW, FDKZVOW KIT FGHD.

– UDIXKSGI QCKIJZGI HINT: D → E

EARLY TO BED, AND EARLY TO RISE, MAKES A MAN HEALTHY, WEALTHY AND WISE.
– BENJAMIN FRANKLIN